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LETTER TO THE EDITOR

Low excitations of the anyon gas

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Abstract. A self-consistent approach to approximate the ground state as well as the lowest excited states of an interacting fermion system is described briefly. The approach is used to study low particle–hole excitations of the many-anyon gas. The excitations found are different and lower than those studied previously (Laughlin R B 1988 *Phys. Rev. Lett.* **60** 2677; Hanna C B, Laughlin R B and Fetter A L 1989 *Phys. Rev.* B **49** 8745) but the energy gap which appears it still logarithmically divergent.

Although the problem of a few anyons is tractable [3–7] and its solution has shed some light on the thermodynamic properties of the dilute anyon gas [8–10], an exact treatment of the many-anyon problem, even in the ideal case of a non-interacting system, is still forbidding and one has to resort to approximation techniques.

In the pioneering work of Laughlin [1] and of Hanna *et al* [2], the Hartree–Fock (HF) approximation was used to study the anyon gas in the Fermi representation. It was shown that in the limit of a large number N of particles and for certain values of the statistical parameter, $\alpha = 1/n$, where n is a non-zero integer, the ground state is composed of Landau orbitals and the n lowest Landau levels are exactly filled. Further, by subtracting the highest occupied energy eigenvalue ϵ_n of an orbital in the nth Landau level from the energy eigenvalue ϵ_{n+1} of an orbital in the next unoccupied level, it was possible to approximate the energy gap which separates the ground state from an excited state corresponding to a single-particle excitation to the $n+1$ level. The result is a logarithmically divergent energy gap. It is then argued [2] that the magnitude of the gap is unphysically large to correspond to a low excitation and the particle and hole states should rather be considered as a natural basis for a higher-order perturbation expansion. They concluded that the low excitations must be soft compressional sound waves.

Logarithmic divergences are not unusual in two-dimensional systems and for a system with a finite number of particles the energy gap would also be finite. However the use of ϵ_{n+1} raises the question of self-consistency of the result. Koopmans' theorem justifies the use of the energy eigenvalue of an orbital, occupied in the ground state for the approximation of the ionization energy. It is ambiguous, however, to use the energy eigenvalue of an orbital which is not occupied in the ground state and deduce results about the excitation energy, especially when the energy gap is so large.

In this work, I describe briefly a self-consistent approach [11, 12], which can be regarded as a generalization of the HF approximation, as it approximates self-consistently the ground state as well as the low excited states of an interacting system of fermions. Using this approach, I investigate the possibility of lower excitations than those found in [1, 2]. In the

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limit of an infinite number of particles, the excitations found can be interpreted as single-particle excitations from occupied orbitals of the nth Landau level to unoccupied orbitals of the same level. However, the energy gap for these excitations is again logarithmically divergent, verifying indirectly the validity of the result in [1,2].

Minimum principles provide us with a powerful tool when an approximate treatment of a physical problem is necessary, in contrast to variational principles in general, which give us information only about the exact solution of the problem. The self-consistent approximation [11,12], is based on a minimum principle [13] referring indirectly to the ground state and the lower excited states of the Hamiltonian $\hat{H}$, which describes a system of N interacting fermions. Thus if V_d is a d-dimensional subspace of the N-particle Hilbert space $\mathcal{H}$ and if $V_d^{(0)}$ is the subspace of $\mathcal{H}$, which is spanned by the d lowest eigenstates of $\hat{H}$, then $\mathrm{Tr}_{V_d}[\hat{H}] \geqslant \mathrm{Tr}_{V_d^{(0)}}[\hat{H}]$, where $\mathrm{Tr}_{V_d}[\hat{H}]$ is the trace of $\hat{H}$ restricted in V_d. The minimum principle guides us to use an appropriate simplified subspace W_d of $\mathcal{H}$ and approximate the whole linear space $V_d^{(0)}$ directly. The subspace W_d of $\mathcal{H}$ should minimize the functional $\mathrm{Tr}_{W_d'}[\hat{H}]$ over all trial subspaces W_d' of $\mathcal{H}$. Let us take for simplicity $d = 2$. In an approximation analogous to the HF, the simplified space W_2 is produced by linear combinations of the N-particle Slater determinant Φ_0 corresponding to the ground state of the interacting fermion system and its lowest excitation. If Φ_0 has the form $\Phi_0 = \det[\psi_1, \psi_2, \ldots, \psi_N]$, with $\psi_1, \psi_2, \ldots, \psi_N$ the orbitals occupied in the ground state, the lowest excitation will normally have the form of a single-particle excitation $\Phi_1 = \det[\psi_1, \psi_2, \ldots, \psi_{N-1}, \psi_{N+1}]$, where the orbital ψ_{N+1} corresponds to the excitation. The orbital space $\{\psi_1, \psi_2, \ldots, \psi_N\}$, together with ψ_{N+1} define the space W_2. In order to minimize the trace $\mathrm{Tr}_{W_2'}[\hat{H}]$ we must vary the orbitals. A system of $N+1$ coupled single-particle equations results. These generalized HF equations describe the ground state and the first excited state of an N-particle system, and they resemble formally, the usual HF equations of some hypothetical $(N+1)$-particle system. The difference is that the various exchange terms are, in this case, multiplied by weights which are different from unity. The form of the generalized HF equations for a Coulomb system can be found in [12], where the method has been applied and tested successfully in the case of simple atoms.

In order to simplify the presentation, the ground and the lower excited states of the anyon gas will be calculated in the Bose limit, i.e., when the statistical parameter is equal to $\alpha = 1$. Similar results are obtained for any value of $\alpha = 1/n$, provided $n \ll N$. The Hamiltonian $\hat{H}$ which describes the system of anyons, at the Bose point is

$$\hat{H} = \frac{\hbar^2}{2M} \int \mathrm{d}^2 r \Psi^\dagger(r)\left[-\nabla^2 - 2\mathrm{i}\hat{A}(r)\cdot\nabla + \hat{A}^2(r)\right]\Psi(r)$$

where $\Psi^\dagger(r)$ and $\Psi(r)$ are second-quantized creation and annihilation field operators, $A(r_1) = \int \mathrm{d}^2 r_2 \Psi^\dagger(r_2) A_{12} \Psi(r_1)$, $r_{12} \equiv r_1 - r_2$, $A_{12} = \frac{\hat{z}\times r_{12}}{r_{12}^2}$ for $|r_{12}| > \lambda$ and $A_{12} = 0$ for $|r_{12}| < \lambda$. In the end of the calculation, the limit $\lambda \to 0$ must be taken.

We consider a large number of N anyons, enclosed in a large circular area of radius L and surface $S = \pi L^2$. Let the ground state have the form $\Phi_0 = \det[\psi_1, \psi_1, \ldots, \psi_N]$. The degeneracy of each level is N, equal to the number of particles and the orbitals ψ_i, $i = 1, \ldots, N$, exactly fill the first Landau level [1,2]. Since we want to study single-particle excitations of the system, we introduce an extra orbital ψ_{N+1}. As the orbitals of the ground state are degenerate, an excited state corresponds to an excitation of a particle from an orbital ψ_i, $i = 1, \ldots, N$, to the orbital Ψ_{N+1} and will have the form $\Phi_n = \det[\psi_1, \ldots, \psi_{n-1}, \psi_{N+1}, \psi_{n+1}, \ldots, \psi_N]$. There are N such Slater determinants Φ_n, $n = 1, \ldots, N$, which, combined with Φ_0, produce the linear space W_{N+1}.

We next transform to dimensionless coordinates:

$$\boldsymbol{x} = \boldsymbol{r}/a \tag{1a}$$

where a is the radius:

$$a = (2\pi\rho)^{-1/2} \tag{1b}$$

and ρ is the density:

$$\rho = (N+1)/\pi L^2. \tag{1c}$$

The orbitals $\psi_i(\boldsymbol{r})$ are transformed to:

$$\phi_i(\boldsymbol{x}) = a\psi_i(a\boldsymbol{x}). \tag{1d}$$

Note that ρ is not the uniform ground state density of the N-anyon system. The density ρ corresponds to the whole space W_{N+1} and its meaning will be apparent shortly. In the same say, a is not the cyclotron radius of the average magnetic field which appears in the ground state of the N-anyon system.

The trace of the Hamiltonian $\hat{H}$ in the linear space W_{N+1} spanned by the determinants Φ_n, $(n = 0, \ldots, N)$, is:

$$\mathrm{Tr}_{W_{N+1}}[\hat{H}] = G_1 + G_2 + G_3 + G_4 \tag{2}$$

where

$$G_1 = \frac{2\pi\rho\hbar^2}{M} N \sum_{i=1}^{N+1} \int \mathrm{d}^2\boldsymbol{x}_1 \phi_i^*(\boldsymbol{x}_1)\left[-\frac{\nabla_1^2}{2}\right]\phi_i(\boldsymbol{x}_1)$$

$$G_2 = \frac{2\pi\rho\hbar^2}{M}(N-1) \sum_{i,j=1}^{N+1} \int\int \mathrm{d}^2\boldsymbol{x}_1 \mathrm{d}^2\boldsymbol{x}_2 \phi_i^*(\boldsymbol{x}_1)\phi_j^*(\boldsymbol{x}_2)\boldsymbol{a}_{12}\cdot(-i\nabla_1)\det\begin{bmatrix}\phi_i(\boldsymbol{x}_1) & \phi_i(\boldsymbol{x}_2)\\ \phi_j(\boldsymbol{x}_1) & \phi_j(\boldsymbol{x}_2)\end{bmatrix}$$

$$G_3 = \frac{2\pi\rho\hbar^2}{M}(N-1) \sum_{i,j=1}^{N+1} \int\int \mathrm{d}^2\boldsymbol{x}_1 \mathrm{d}^2\boldsymbol{x}_2 \phi_i^*(\boldsymbol{x}_1)\phi_j^*(\boldsymbol{x}_2)\frac{\alpha_{12}^2}{2}\det\begin{bmatrix}\phi_i(\boldsymbol{x}_1) & \phi_i(\boldsymbol{x}_2)\\ \phi_j(\boldsymbol{x}_1) & \phi_j(\boldsymbol{x}_2)\end{bmatrix}$$

$$G_4 = \frac{2\pi\rho\hbar^2}{M}(N-2) \sum_{i,j,k=1}^{N+1} \int\int\int \mathrm{d}^2\boldsymbol{x}_1 \mathrm{d}^2\boldsymbol{x}_2 \mathrm{d}^2\boldsymbol{x}_3 \phi_i^*(\boldsymbol{x}_1)\phi_j^*(\boldsymbol{x}_2)\phi_k^*(\boldsymbol{x}_3)\frac{\alpha_{12}\alpha_{13}}{2}$$
$$\times \det\begin{bmatrix}\phi_i(\boldsymbol{x}_1) & \phi_i(\boldsymbol{x}_2) & \phi_i(\boldsymbol{x}_3)\\ \phi_j(\boldsymbol{x}_1) & \phi_j(\boldsymbol{x}_2) & \phi_j(\boldsymbol{x}_3)\\ \phi_k(\boldsymbol{x}_1) & \phi_k(\boldsymbol{x}_2) & \phi_k(\boldsymbol{x}_3)\end{bmatrix}$$

and $\alpha_{12} = \frac{\hat{z}\times\boldsymbol{x}_{12}}{x_{12}^2}$ for $|\boldsymbol{x}_{12}| > \lambda$ while it vanishes for $|\boldsymbol{x}_{12}| < \lambda$. The coefficient N in G_1 is the number of Slater determinants Φ_n in which a given orbital ϕ_i is occupied. Similarly the coefficients $N-1$ and $N-2$ which appear in G_2, G_3 and G_4 denote the number of determinants which occupy a given pair of orbitals $\{\phi_i, \phi_j\}$ or a triplet $\{\phi_i, \phi_j, \phi_k\}$. We observe that the orbitals ϕ_i, $i = 1, \ldots, N+1$, are equivalent in the functional form of the trace, in the sense that an interchange of any two orbitals leaves the trace unchanged. This

is due to the fact that W_{N+1} is produced by all the N-particle Slater determinants which can be constructed out of the $N+1$ orbitals.

Assuming that the particular set of orbitals minimizes the trace $\mathrm{Tr}_{W_{N+1}}[\hat{H}]$, we vary an orbital in order to derive the generalized HF equations. Dividing by $2\pi\rho\hbar^2 N/M$, the equations which result from the variation are:

$$\left[-\frac{\nabla_1^2}{2}\right]\phi_i(x_1)+\left(1-\frac{1}{N}\right)\sum_{j=1}^{N+1}\int \mathrm{d}^2x_2\phi_j^*(x_2)\left[-i\alpha_{12}\cdot\nabla_1-i\alpha_{21}\cdot\nabla_2\right]\det\begin{bmatrix}\phi_i(x_1) & \phi_i(x_2)\\ \phi_j(x_1) & \phi_j(x_2)\end{bmatrix}$$

$$+\left(1-\frac{1}{N}\right)\sum_{j=1}^{N+1}\int \mathrm{d}^2x_2\phi_j^*(x_2)\alpha_{12}^2\det\begin{bmatrix}\phi_i(x_1) & \phi_i(x_2)\\ \phi_j(x_1) & \phi_j(x_2)\end{bmatrix}$$

$$+\left(1-\frac{1}{N}\right)\sum_{j,k=1}^{N+1}\int\int \mathrm{d}^2x_2\mathrm{d}^2x_3\phi_j^*(x_2)\phi_k^*(x_3)\frac{1}{2}\left[\alpha_{12}\cdot\alpha_{13}+\alpha_{21}\cdot\alpha_{23}+\alpha_{32}\cdot\alpha_{31}\right]$$

$$\times\det\begin{bmatrix}\phi_i(x_1) & \phi_i(x_2) & \phi_i(x_3)\\ \phi_j(x_1) & \phi_j(x_2) & \phi_j(x_3)\\ \phi_k(x_1) & \phi_k(x_2) & \phi_k(x_3)\end{bmatrix}=\epsilon_i\phi_i(x_1).$$

The resulting equations have a similar form as the usual HF equations for the ground state of a hypothetical $(N+1)$-particle system. The only difference are the coefficients $(1-\frac{1}{N})$ and $(1-\frac{2}{N})$ which multiply the exchange terms of the HF Hamiltonian but do not affect the direct Hartree term. We know from [1,2] that the Hartree term gives the qualitative description of the system and in the limit of a large number of particles, the exchange terms do not alter this picture but only correct the energy eigenvalues of the orbitals and the total energy of the system. Therefore, the idea of the hypothetical $(N+1)$-particle system can be useful for the solution of the generalized HF equations. Suppose that $\phi_1, \ldots, \phi_{N+1}$ are the orbitals which constitute the self-consistent solution of the generalized HF equations. Then $\Phi = \det[\phi_1, \ldots, \phi_{N+1}]$ will be the $(N+1)$-particle Slater determinant describing the ground state of the hypothetical system. The density $\sum_{j=1}^{N+1}|\phi_j(x)|^2$ of Φ will be uniform since the generalized HF Hamiltonian is translationally invariant. The interpretation of the density ρ, equation (1c), is apparent, it is the ground state density of the hypothetical system, in terms of the unscaled orbitals $\rho = \sum_{j=1}^{N+1}|\psi_j(r)|^2$. From the scale transformation $\sum_{j=1}^{N+1}|\phi_j(x)|^2 = \frac{1}{2\pi}$.

The Hartree approximation to the problem, apart from a constant term, is given by $H_1(x)\phi_1(x) = \mu_i\phi_i(x)$, $H_1(x) = \frac{1}{2}[-i\nabla + \alpha(x)]^2$ and $\alpha(x) = 1/2z \times x$. Each charged particle of the hypothetical system interacts with the uniform magnetic field produced by the particles. The radius a, which was defined in the scale transformation, is the cyclotron radius of this average magnetic field. The eigenstates of H_1 are orbitals which fill the Landau levels. The degeneracy of each Landau level, using equation (1a–c) is easily found to be equal to the total number of orbitals. The $N+1$ orbitals therefore exactly fill the first Landau level. Hence the orbital ϕ_{N+1}, which we introduced to study the excitations from the ground state, belongs to the first Landau level. In the limit $N \to \infty$, we may substitute the density matrix $\rho(x_1, x_2) = \sum_{j=1}^{N+1}\phi_j(x_1)\phi_j^*(x_2)$, with the projection operator $\Pi_0(1,2)$ for the first Landau level [2]. One can then show, following Hanna *et al* [2] that each term of the generalized HF Hamiltonian is either equal to a constant or can be written as a linear combination of projection operators Π_n for the nth Landau level.

We conclude that the self-consistent eigenstates of the generalized HF Hamiltonian are the orbitals which fill the first Landau level.

Let us suppose that the number of particles is large but finite. As the orbitals are known, we can construct the Slater determinants Φ_n, $n = 0, \ldots, N$. We must remember that the orbitals which build Φ_0 are not identical to the orbitals which are self-consistent solutions of the HF equations for the ground state of the N-particle problem. They are scaled differently and they are also changed slightly by the variation. In the limit of an infinite number of particles the two orbital sets coincide.

The approximate ground state $\Psi^{(0)}$ will be given by a linear combination of all the Slater determinants $\Psi^{(0)} = \sum_{m=1}^{N+1} c_m^{(0)} \Phi_m$, and not by Φ_0 along. Similarly, the approximate excited states will be given by $\Psi^{(n)} = \sum_{m=1}^{N+1} c_m^{(n)} \Phi_m$. The coefficients $c_m^{(n)}$, $n = 0, \ldots, N$ are the eigenstates of the matrix $\langle \Phi_n | \hat{H} | \Phi_m \rangle$. Its eigenvalues give the approximate ground and excited state energies.

If we assume that in the limit $N \to \infty$ the weight $c_0^{(0)}$ of Φ_0 in the linear combination for the ground state approaches unity, while the weights $c_0^{(n)}$ of Φ_0 in the linear combinations for the excited states vanish, then the excitations we have found may be interpreted as excitations from an unoccupied orbital of the first Landau level to an unoccupied orbital of the same level.

The proper way to find the ground and the excited state energies is by diagonalizing the matrix $\langle \Phi_n | \hat{H} | \Phi_m \rangle$. However we shall take a shortcut. Since we know the orbitals we can calculate explicitly the terms $G_1, \ldots, G_4$, equation (2), and find the trace $\mathrm{Tr}_{W_{N+1}}[\hat{H}]$. Assuming the excited states we study are degenerate with energy E_1, the trace $\mathrm{Tr}_{W_{N+1}}[H]$, gives us the sum of the ground state energy E_0 and E_1, with weight N. The result in units $2h^2N/(ML^2)$ is equal to $E_0 + NE_1 = \frac{1}{4}(N+1)^2 + \frac{1}{4}(N+1)^2[\ln(N+1)+\gamma]/(N+2)$, where $\gamma = 0.57721566...$ is Euler's constant. The accuracy of the results is of the order of unity which is satisfactory compared with the large values of N and $\ln N$. In the calculation the different coefficients N, $(N-1)$ and $(N-2)$ in front of G_1, $G_2 - G_3$ and G_4 distinguish our case from the ground state of some real $(N+1)$-anyon system. If the above coefficients were equal, the logarithm in the result would not appear. Using the result for the ground state energy from [1, 2], $E_0 = \frac{1}{4}N$, we easily find that $E_1 = \frac{1}{4}N + \frac{1}{2}E_N$, where E_N coincides with the corresponding quantity defined in [2], $E_N \equiv \frac{1}{2}(\ln N + \gamma)$. The energy gap therefore is equal to $\Delta E = \frac{1}{2}E_N$.

It is somewhat surprising to see that the energy gap for a single-particle excitation from an occupied orbital of the first Landau level, to an unoccupied orbital of the same level, diverges logarithmically, in a similar way as with the excitation of a particle to the next unoccupied level, although the value of the gap is smaller, by exactly half. The logarithmic nature of the gap suggests that the particle and hole excitations are charged vortices [1, 2, 14]. The magnitude of the gap, however, is still very large and the excited states we have found should be considered as a basis for a perturbation expansion [2].

Finally let me note that the present excitations appear only when we start with a finite number of particles and a finite area of the system, then derive the equations to study the excitations and only in the end let $N, L \to \infty$. This leads me to believe that these excitations are connected with the boundary of the system.

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